1)

c

a) Let $\phi: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a scalar function and let $\in \mathbb{R}$. Suppose that $\phi(x,y,z) = c$ defines a surface S in $\mathbb{R}^3$. Also suppose that $\underline{\nabla}\phi$ exists and non-zero at each point of $\mathbb{R}^3$. Let $P \equiv (x_0, y_0, z_0)$ be a point of S. Show that the vector equation of the tangent plane to the surface S at P is

$$\underline{r} \cdot (\underline{\nabla}\phi)_P = \left(x_0\underline{i} + y_0\underline{j} + z_0\underline{k}\right) \cdot (\underline{\nabla}\phi)_P$$

where $\underline{r}$ is the position vector of an arbitrary point on the tangent plane. Hence show that a Cartesian equation of the that tangential plane is

$$x\left(\frac{\partial\phi}{\partial x}\right)_P + y\left(\frac{\partial\phi}{\partial y}\right)_P + z\left(\frac{\partial\phi}{\partial z}\right)_P = \left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z}\right)_P.$$

b) Consider the surface $x^2 + 2y^2 + z^2 = 7$. Find a Cartesian equation of the tangent plane Π, to the surface at $(1,-1,2)$ and find the point of intersection of the straight line l : $\underline{r} = \underline{i} + \lambda\underline{j}$ and the plane Π.

c) Find a Cartesian equation of a plane which touches the surface $x^2 + z^2 = 4$ and goes through the point $(1, 2, -3)$.

2)

a) Let $\underline{F}$ be defined as,

$$\underline{F}(x,y,z) = (2xy^3 + zSin(y))\underline{i} + (3y^2x^2 + xzCos(y))\underline{j} + xSin(y)\underline{k}$$

for any $(x,y,z) \in \mathbb{R}^3$.

(i) Show that $\underline{F}$ is a conservative vector field.
(ii) Find the potential function of $\underline{F}$.

b) Suppose that $T(x,y,z) = x^2yz$ gives the temperature at any $(x,y,z) \in \mathbb{R}^3$. Suppose a particle moves with velocity vector $\underline{i} - 2\underline{j} + 3\underline{k}$. What is the rate of increase of temperature the particle experiences, when it passes through the point $(4, 1, -1)$?

3)

a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 2y^3 - 12x^3 - 21y^2 + 36xy + 7$ for each $(x,y) \in \mathbb{R}^2$.

(i) Locate all the critical points of f.
(ii) Does f have any saddle points? Justify your answer.
(iii) Show that f attains a local maximum at $(1,1)$.
(iv) Is $f(1,1)$ an absolute maximum? Justify your answer.

b) Find the minimum distance from the point $(1,-1,1)$ to the plane $x + 2y + 3z = 1$.
(Hint: Consider the square of the distance.)

4) Let $\underline{F}(x,y,z) = (x+y)\underline{i} + (y^2+z)\underline{j} + xz\underline{k}$ for each $(x,y,z) \in \mathbb{R}^3$.

a) Find $\int_C \underline{F} \cdot d\,\underline{r}$ where the path of integration C specified by $z = 1 + x^2 + y^2$ and $x = 2y$ from $(0,0,1)$ to $(-2,-1,6)$.

b) Calculate the volume integral $\iiint_V (\underline{\nabla} \cdot \overline{F})\,dv$, where V is the volume of the finite cylinder specified by $x^2 + y^2 \leq 4$ and $3 \leq z \leq 8$.

1) Let A, B, C and D be four distinct points such that $\overrightarrow{OA} = 3\underline{i} + 2\underline{j} + 3\underline{k}$, $\overrightarrow{OB} = 2\underline{i} + \underline{j}$, $\overrightarrow{OC} = 4\underline{j} + 4\underline{k}$ and $\overrightarrow{OD} = 7\underline{i} + 4\underline{k}$, where O is the origin.

a) Let α be the plane passing through the points A, B and C. Find the equation of α in scalar form.

b) Let l be the straight line passing through A and the midpoint of the line segment BC. Find the equation of l in scalar form.

c) Let β be the plane passing through the point D and is perpendicular to the line l. Find the equation of β in scalar form.

d) Calculate the angle between α and β.

2) a) The position $S(t)$ of a particle moving in an accelerator at time t is given by,

$$S(t) = \sin t\,\underline{i} + t\cos t\,\underline{j} + (t^2 + e^t)\underline{k}.$$

Find the velocity and acceleration of the particle at $t = \frac{\pi}{2}$.

b) Let φ, ϕ be two scalar functions of independent variables x, y and z. Prove the followings.

i) $\nabla(\varphi + \phi) = \nabla(\varphi) + \nabla(\phi)$.

ii) $\nabla(\varphi\phi) = \varphi\nabla(\phi) + \phi\nabla(\varphi)$.

c) Quantity produced (Q) of a commodity is given by,

$$Q = 4B + R^3L^{12}$$

where B, R, L are continuous parameters on OX, OY, OZ respectively. Find,

i) ∇Q when $B = R = L = 1$.

ii) the directional derivative of Q in the direction $4\underline{i} + 3\underline{j} + 12\underline{k}$ at $B = R = L = 1$.

3) a) Let ϕ be the surface of the ellipsoid given by $\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{4}=1$ and φ be the surface $xyz = C$, where C is a positive constant. Let (x_0, y_0, z_0) lies on both ϕ and φ.

i) Find $\underline{a}$, $\underline{b}$ normal vectors to ϕ, φ respectively, at (x_0, y_0, z_0).

ii) Suppose there exists a real number λ such that $\underline{a} = \lambda \underline{b}$. Show that $\lambda C = \frac{2}{3}$ and $C = \frac{8}{\sqrt{3}}$.

b) Let $\underline{F}$ be a vector field defined by,

$$\underline{F}(x, y, z) = (1 + \sin y)\underline{i} + (x\cos y + z^2)\underline{j} + (2yz + e^z)\underline{k}.$$

i) Show that $\underline{F}$ is a conservative field.

ii) Find the scalar potential ϕ.

4) a) Evaluate $\int_1^2 \underline{F}(t)dt$ where $\underline{F}(t) = (t^4 + t + 2)\underline{i} + (e^t + \sin \pi t + t)\underline{j} + te^{t^2}\underline{k}$.

b) Let $\underline{F}(x, y, z) = (xy + 2)\underline{i} + (x^3z - yz^2)\underline{j} + (x + y + 3z)\underline{k}$ and C be the path from P to S, defined by straight lines PQ, QR, RS where $P \equiv (2,0,0), Q \equiv (2,1,1), R \equiv (1,1,1), S \equiv (2,2,2)$. Evaluate $\int_C \underline{F}(x, y, z) \cdot d\underline{r}$ where $d\underline{r} = dx\underline{i} + dy\underline{j} + dz\underline{k}$.

c) Using Divergence Theorem (Gauss's Theorem) compute $\iint_S \underline{F} \cdot \underline{n}\, ds$ where, $\underline{F}(x, y, z) = (x + y + z)\underline{i} + (x + 2y + 4z)\underline{j} + (x - 3y + 9z)\underline{k}$, S is the surface of unit sphere $x^2 + y^2 + z^2 = 1$ and $\underline{n}$ is the unit vector normal to ds.

1)

Let $\underline{a} = \underline{i} - \underline{j}$, $\underline{b} = \underline{i} + \underline{j} + \underline{k}$ and $\underline{c} = 4\underline{i} + 2\underline{j} + 3\underline{k}$ be vectors; O is the origin and suppose $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, and $\overrightarrow{OC} = \underline{c}$.

a) Find the area of the triangle ABC.
b) Show that the points $O, A, B, and\ C$ are on a plane.
c) Find equation of the plane go through the points $O, A, B, and\ C$.

2)

a) A particle moves along a curve $x = 5t + 1$, $y = 5t^3 + t - 1$, $z = 4t$ where t is the time. Find the components of velocity and acceleration at time $t = 1$ in the direction $\underline{i} + 2\underline{j} + 2\underline{k}$

b) Find the equation for the tangent plane to the surface $2xy - z^3 + 4x = \cancel{10}\ -2$ at the point $(1,1,2)$.

c) Determine the directional derivative of $\phi = xyz^3 + x^3 y + z$ at the point $(1,2,1)$ in the direction of the vector $\underline{i} + 2\underline{j}$.

3)

a) Prove that $F(x,y,z) = (y^2 \cos x + z^3)\underline{i} + (2y \sin x - 4)\underline{j} + (3xz^2 + 2)\underline{k}$ is a conservative field. find its scalar potential $\phi(x,y,z)$.

b) Find a unit normal to the surface $x^2 y^2 + 2yz + 5 \sin x = 4$ at the point $(0,-1,2)$.

4)

If $\underline{A} = (x^2 - y^2)\underline{i} + 2xy\underline{j}$

a) Find $\underline{\nabla} \times \underline{A}$

b) Evaluate $\iint_s (\underline{\nabla} \times \underline{A}) \cdot \underline{ds}$ over a rectangular surface in the xy plane bounded by $x = 0$, $x = a$ $y = 0$ and $y = b$

c) Evaluate $\oint_c \underline{A} \cdot \underline{dr}$ around above rectangle and verify stoke's theorem.

1. (a) Let A, B, C and D be points with coordinates $(4,5,1), (0,-1,-1), (3,9,4)$ and $(-4,4,4)$. Use **vector methods** for the following problems.

   i. Find the vector equation and cartesian equation of the straight line which passes through A and parallel to $\overrightarrow{BC}$.

   ii. Find the area of the triangle BCD.

   iii. Show that A, B, C and D are coplanar.

   iv. Find the vector equation and cartesian equation of the plane $ABCD$.

   (b) Let $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ be non zero vectors. Show that,

   i. $[\, \underset{\sim}{a}+\underset{\sim}{b}\,, \underset{\sim}{b}+\underset{\sim}{c}\,, \underset{\sim}{c}+\underset{\sim}{a}\,] = 2[\, \underset{\sim}{a}\,, \underset{\sim}{b}\,, \underset{\sim}{c}\,]$

   ii. $\underset{\sim}{i} \times (\underset{\sim}{a} \times \underset{\sim}{i}) + \underset{\sim}{j} \times (\underset{\sim}{a} \times \underset{\sim}{j}) + \underset{\sim}{k} \times (\underset{\sim}{a} \times \underset{\sim}{k}) = 2\underset{\sim}{a}$

2. (a) The position $S(t)$ of a particle moving in an accelerator at time t is given by,

$$S(t) = \sin 2t\ \underset{\sim}{i} + \cos t\ \underset{\sim}{j} + e^t\ \underset{\sim}{k}.$$

   Find the velocity and acceleration of the particle at $t = \dfrac{\pi}{4}$.

   (b) Find the cartesian equation of the tangent plane to the surface $x^2z - y^3z^5 + xy = 9$, at the point $P \equiv (1,2,-1)$.

   (c) Find the directional derivative of the function $\phi(x,y,z) = x^2 + y^2 + z^2$ at the point $(2,2,1)$ in the direction of $\langle 2,2,1 \rangle$.

3. Consider the vector field $\underset{\sim}{A} = 3x^2y\underset{\sim}{i} + (x^3 - 2yz^2)\underset{\sim}{j} + (3z^2 - 2y^2z)\underset{\sim}{k}$.

   (a) Show that $\underset{\sim}{A}$ is irrotational and not solenoidal.

   (b) Find the scalar potential of $\underset{\sim}{A}$.

   (c) Evaluate $\int_C \underset{\sim}{A}.d\underset{\sim}{r}$, where C is the straight line from $(0,4,5)$ to $(1,1,2)$.

4. (a) Let $\underline{F} = xy\underline{i} + y^2\underline{j} + xz\underline{k}$. Find $\int_C \underline{F}.d\underline{r}$, where C is defined as the semicircle $x^2+z^2 = 1, y = 3$ from $(1,3,0)$ to $(-1,3,0)$ through $(0,3,1)$.

(b) Evaluate

$$\iint_R (y^2 - xy)dydx$$

, where R is the region in the XY plane bounded by the $x-$ axis, the parabola $y = x^2$, and the line $x = 2$.

(c) Use Green's Theorem to evaluate the integral

$$\int_C e^{-x}(\cos y\ dx - \sin y\ dy)$$

, where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $\left(\pi, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right)$.

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